Math 10A
Midterm I; Friday, 7/6/2018
Time: 2:10 PM **Name:** ____________________
Instructor: Roy Zhao **Student ID:** ____________________

- **DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!**
- Do all problems as best as you can. The exam is 80 minutes long. You may not leave during the last 30 minutes of the exam.
- Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. **No extra sheets of paper can be submitted with this exam!**
- The exam is closed notes and book, which means: **no class notes, no review notes, no textbooks, and not other materials can be used during the exam.** You can only use your cheat sheet. The cheat sheet is one side of one regular 8×11 sheet, handwritten.
- **NO CALCULATORS ARE ALLOWED DURING THE EXAM!**
- Justify all your answers, include all intermediate steps and calculations, and box your answers.

1. (16 points) Calculate the following limits and derivatives.

(a) (3 points) $\lim_{x \to 10} \frac{x^2 + 5}{x - 5} =$

Solution: We can plug in $x = 10$ to get $\frac{105}{5} = \boxed{21}$.

(b) (3 points) $\frac{d}{dx}(x^4 + x^2 + 1) =$

Solution: Using the power rule, it is $4x^3 + 2x$.

(c) (5 points) $\frac{d}{dx}(\sin(e^x)) =$

Solution: Using Chain Rule, it is $\cos(e^x) \cdot e^x$.

(d) (5 points) $\lim_{x \to 0} \frac{e^x - \cos(x) - x}{x^2} =$

Solution: Repeatedly using L'Hopital's Rule gives us

$$\lim_{x \to 0} \frac{e^x - \cos(x) - x}{x^2} = \lim_{x \to 0} \frac{e^x + \sin(x) - 1}{2x} = \lim_{x \to 0} \frac{e^x + \cos(x)}{2} = 1.$$

2. (25 points) Let $f(x) = xe^{x+1}$.

(a) (7 points) Calculate the derivatives $f'(x)$ and $f''(x)$ (factor if necessary) and find the zeros of $f(x), f'(x)$ and $f''(x)$.

Solution: Since $e^x > 0$, we have $xe^x = 0 \implies x = 0$. Then $f'(x) = xe^{x+1} + e^{x+1} = (x+1)e^{x+1}$ so $f'(x) = 0 \implies x = -1$ and $f''(x) = (x+1)e^{x+1} + e^{x+1} = (x+2)e^{x+1} \implies x = -2$.

(b) (18 points) Sketch the graph of $f(x)$. The graph must clearly show: increase/decrease, concavity, asymptotes. Include your calculations to justify how you found these features. (Note: $2/e \approx 0.75$)

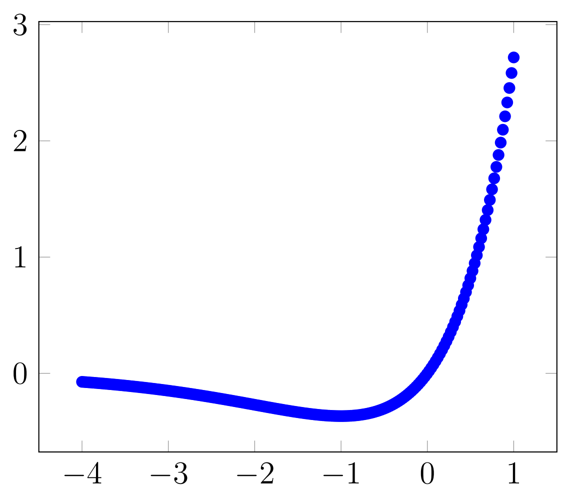
Solution: The function is negative left of 0 and positive to the right.

f' is negative left of -1 and positive to the right.

f'' is negative left of -2 and positive to the right.

The domain is all of $\mathbb{R}$.

$\lim_{x\to-\infty} xe^{x+1} = 0 = \lim_{x\to-\infty} \frac{x}{e^{-x-1}}$. Plugging in $-\infty$ gives ∞/∞ so we use L'Hopitals to get $\lim_{x\to-\infty} \frac{1}{-e^{-x-1}} = 0$. Then $\lim_{x\to\infty} xe^{x+1} = \infty$.



3. (15 points) A farmer wishes to divide his farmland along a straight river into 6 smaller identical rectangular plots by using one fence parallel to the river and 7 fences perpendicular to it. If he has 14 miles of fencing, what is the maximum area he can enclose?

Solution: Let the perpendicular sides have side length s and the parallel fence have side length w. Then the total length of fencing is $7s + w$ and $7s + w = 14$. We want to maximize the total area which is sw. Putting it all in terms of s gives us $w = 14 - 7s$ or $s(14 - 7s) = 14s - 7s^2$ as the function we want to maximize.

The end points are $s = 0$ and $w = 0$ or $w = 0 \implies s = 2$. Then the critical points are when $(14s - 7s^2)' = 14 - 14s = 0$ or $s = 1$. Plugging in $s = 0, 1, 2$ into the total area of $14s - 7s^2$ tells us that the maximum area he can enclose is when $s = 1$ and $w = 14 - 7s = 7$ so the area is $\boxed{7m^2}$ because plugging in 0 and 2 give us $0m^2$.

4. (18 points) A conical cup that is $6cm$ wide at the top and $6cm$ tall is filled with water is punctured at the bottom and water is coming out at a rate of $\pi cm^3/s$. How fast is the height of the water changing when the height is $2cm$? (Note: The volume of a cone with radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$)

Solution: If the height of the water is h, then the radius of the cone formed by the water would be $3/5h$ and so the volume of the water cone is $V = \pi/3(3/6h)^2 \cdot h = \dfrac{3\pi h^3}{36}$. Taking the derivative of both sides gives

$$V' = \frac{9\pi h^2 h'}{36}$$

and plugging in $-\pi$ for V' and 2 for h gives

$$-\pi = \frac{9\pi 4 \cdot h'}{36} \implies h' = -1cm/s.$$

5. (16 points) (a) (8 points) Use a second order Taylor polynomial of e^x around $x = 0$ to approximate $e^1 = e$.

Solution: The second order Taylor series around $x = 0$ is $f(x) = f(0)+f'(0)x+f''(0)/2!x^2 = 1 + x + x^2/2$. To approximate $e^1 = e$, we plug in $x = 1$ to get $1 + 1 + 1/2 = 2.5$.

(b) (8 points) Use one iteration of Newton's method to find the critical point of $\sin(x)-x^2$ starting with a guess of $x_0 = 0$.

Solution: The function we want to find a 0 of is the derivative which is $f(x) = \cos(x) - 2x$. By the formula, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{\cos(0) - 2 \cdot 0}{-\sin(0) - 2} = 0 - \frac{1 - 0}{-0 - 2} = \frac{1}{2}.$$

6. (10 points) Circle True or False. (1 point for correct answer, 0 if incorrect)

(a) True **FALSE** If f is defined at $x = 0$, then $\lim_{x\to 0} f(x) = f(0)$.

Solution: It is possible for the function to have a discontinuity at $x = 0$.

(b) True **FALSE** If the function f is not invertible, then there is no x such that $f(x) = 7$.

Solution: The function $f(x) = x^2$ is not invertible but $f(x) = 7$ has a solution.

(c) **TRUE** False The range of an invertible function f is the domain of the inverse f^{-1}

(d) **TRUE** False If f is continuous on $[0, 2]$, then $\lim_{x\to 1} f(x) = f(1)$.

(e) True **FALSE** If the derivative of a function f is negative at $x = c$, then $f(c) < 0$.

(f) True **FALSE** If the derivative of f is increasing, then f is increasing as well.

(g) True **FALSE** If $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$.

(h) **TRUE** False If $T(x)$ is the third order Taylor polynomial of $f(x)$ centered at $x = a$, then $f(a) = T(a)$.

(i) True **FALSE** Newton's method always converges to the zero of a function.

(j) True **FALSE** If we are using Newton's method to approximate $\sqrt{17}$ with an initial guess of $x_0 = 4$, then we apply Newton's method to the function $f(x) = \sqrt{x}$.

Solution: We apply Newton's method to $x^2 - 17$.